

RIEMANNIAN SUBMERSIONS FROM ALMOST CONTACT METRIC MANIFOLDS

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ABSTRACT. In this paper we obtain the structure equation of a contact-complex Riemannian submersion and give some applications of this equation in the study of almost cosymplectic manifolds with Kähler fibres.

1. INTRODUCTION

The Riemannian submersions are of great interest both in mathematics and physics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories [3, 4, 10, 18, 19, 22, 29, 33], they being intensively studied for different ambient spaces by several authors (see e.g. [1, 6, 7, 9, 11, 12, 15, 16, 23, 26, 27, 28, 30, 31, 32, 36]). In this paper we study the so-called contact-complex Riemannian submersions, i.e. Riemannian submersions from an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ onto an almost Hermitian manifold (N, J, g') , which are (φ, J) -holomorphic mappings (see [17]). The paper is organized as follows. The second section contains some preliminaries on almost contact metric manifolds and Riemannian submersions. In the next section, some results concerning the contact-complex submersions are given; among other, we discuss the transference of Gray-type curvature conditions from the total space to the fibres and to the base space. In the 4th section we obtain the structure equation of a contact-complex Riemannian submersion. This equation is used in the last section in the study of almost cosymplectic manifolds with Kähler fibres. In particular, we prove that the fundamental 2-form of such kind of manifolds is harmonic.

2. PRELIMINARIES

As usual, all manifolds and maps involved are assumed to be of class C^∞ . Furthermore, we denote by $\Gamma(E)$ the space of the sections of a vector bundle E .

Let M be a differentiable manifold. An almost contact structure on M , denoted by (φ, ξ, η) , consists in a $(1,1)$ -tensor field φ , a non-singular vector field ξ , and a non-singular 1-form η that verify:

$$(2.1) \quad \varphi^2 = -I + \eta \otimes \xi$$

and

$$(2.2) \quad \eta(\xi) = 1,$$

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where I stands for the identity endomorphism of the fiber bundle TM . It follows (see e.g. [2, 8]) that the manifold has odd dimension and there holds $\varphi\xi = 0$ and $\eta \circ \varphi = 0$, as well. Moreover, (M, φ, ξ, η) is called an almost contact manifold.

A Riemannian metric g on M is said to be adapted to the almost contact structure (φ, ξ, η) if

$$(2.3) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any sections $X, Y \in \Gamma(TM)$. The setting (φ, ξ, η, g) is called an almost contact metric structure on M , while M is an almost contact metric manifold. It follows from (2.1)-(2.3) that the relation: $\eta(X) = g(X, \xi)$ holds for any section $X \in \Gamma(TM)$. Define next the fundamental 2-form Φ of the almost contact metric manifold M by the analogous to the almost Hermite geometry formula

$$(2.4) \quad \Phi(X, Y) = g(X, \varphi Y)$$

for any sections of the tangent bundle $X, Y \in \Gamma(TM)$. Basic examples and features of those structures will be referred from the book of Blair [2].

Let $\pi : M \rightarrow N$ be a surjective differentiable map between two smooth manifolds M and N . We call it a submersion if its rank is constant, equal to the dimension of the target manifold N .

Let now M and N be Riemannian manifolds with Riemannian metrics g and g' respectively. Write $F_x = \pi^{-1}(x')$ for the fibre in $x \in M$ of the map $\pi : M \rightarrow N$, where $\pi(x) = x'$. The tangent vectors to the fibre belong to the kernel of the linear map $\pi_* = d\pi$; one calls them vertical. The distribution of the vertical vectors, \mathcal{V} , is well defined (with constant rank) and $\mathcal{V} = \text{Ker } d\pi$. It follows that \mathcal{V} is integrable and the (connected components of the) fibres are its integral manifolds (of maximal rank). The orthogonal complement of \mathcal{V} in TM with respect to the Riemannian metric g is the horizontal distribution, denoted by \mathcal{H} . The submersion $\pi : (M, g) \rightarrow (N, g')$ is a Riemannian submersion if $d\pi_x : \mathcal{H}_x \rightarrow T_{x'}N$ becomes a linear isometry, for any $x \in M$.

If an horizontal vector field (a section in the horizontal distribution \mathcal{H}) is π -related (so that projectable) to a vector field on the base manifold N , one calls it basic. It follows from the definition of the Riemannian submersion that

$$(2.5) \quad g(X, Y) = g'(X', Y') \circ \pi$$

if X and Y are basic vector fields, π -related to $X', Y' \in \Gamma(TN)$, respectively.

The projections in TM on \mathcal{V} and \mathcal{H} will be respectively denoted by v , and h , as well. If ∇ and ∇' are the Levi-Civita connections of the Riemannian metrics on M , N , then we have the following (see [24]).

Proposition 2.1. *Let $\pi : (M, g) \rightarrow (N', g')$ be a Riemannian submersion. If X, Y are basic vector fields π -related to X', Y' on N , then:*

- i. $h[X, Y]$ is basic, π -related to $[X', Y']$.
- ii. $h(\nabla_X Y)$ is basic, π -related to $\nabla'_{X'} Y'$.
- iii. $[X, V]$ is vertical for any $V \in \Gamma(\mathcal{V})$.

The fundamental tensors of a submersion were defined by O'Neill. They are (1, 2)-tensors on M , given by the formulae:

$$(2.6) \quad T_{\tilde{X}} \tilde{Y} = h \nabla_{v \tilde{X}} v \tilde{Y} + v \nabla_{v \tilde{X}} h \tilde{Y},$$

$$(2.7) \quad A_{\tilde{X}} \tilde{Y} = v \nabla_{h \tilde{X}} h \tilde{Y} + h \nabla_{h \tilde{X}} v \tilde{Y},$$

where \tilde{X}, \tilde{Y} are vector fields on M . One gets immediately:

$$(2.8) \quad T_U W = T_W U, \quad \forall U, W \in \Gamma(V)$$

and

$$(2.9) \quad A_X Y = -A_Y X = \frac{1}{2}v[X, Y], \quad \forall X, Y \in \Gamma(H).$$

For the sake of simplicity, U, V, W etc. will stand for vertical vectors, X, Y, Z etc. for horizontal vectors or vector fields, respectively.

This paper will consider Riemannian submersions where the total space M is an almost contact metric manifold and the base space N is an almost Hermitian manifold with an almost complex structure J compatible to g' . Let Ω be the fundamental 2-form of (N, g', J) so that $\Omega(X', Y') = g'(X', JY')$. Basic properties of almost Hermitian manifolds are found e.g. in the fundamental work of Kobayashi and Nomizu [20].

Recall the definition of (φ, J) -holomorphic mappings from an almost contact manifold to an almost complex manifold (see [17]).

Definition 2.2. Let (M, φ, ξ, η) be almost contact and (N, J) be almost complex manifolds, respectively. The map $\pi : M \rightarrow N$ is (φ, J) -holomorphic if $J \circ d\pi = d\pi \circ \varphi$.

Now let consider Riemannian structures.

Definition 2.3. Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold and (N, J, g') be an almost Hermitian manifold. A Riemannian submersion $\pi : M \rightarrow N$ is a contact-complex (Riemannian submersion) if it is (φ, J) -holomorphic, as well.

Next, we recall some examples (see [13]).

Example 2.4. For (M, g_1) and (N, g_2) Riemannian manifolds, take the product $M \times N$. Its tangent bundle naturally splits as $T(M \times N) = TM \oplus TN$. For $(x, y) \in M \times N$ one defines the symmetric bilinear forms $g_1 \oplus g_2$ on all $T_{(x,y)}(M \times N) = T_x M \oplus T_y N$ as

$$(2.10) \quad g = \pi_1^* g_1 + \pi_2^* g_2$$

where $\pi_1 : M \times N \rightarrow M$ and $\pi_2 : M \times N \rightarrow N$ are the natural projections. One gets a Riemannian tensor on $M \times N$ and a Riemannian submersion from $M \times N$ to M (the mapping π_1).

Example 2.5. R. Bishop and B. O'Neill introduced the warped product of Riemannian manifolds, which varies the metric on some factor, e.g. $\tilde{g}_2 = f(x)g_2$, where f is a positive function on N and

$$g = \pi_1^* g_1 + (f \circ \pi_1) \pi_2^* g_2.$$

The metric on the fibres depends upon the value of the map in the base point of them. It holds that $\pi_1 : (N \times_f F, \tilde{g}) \rightarrow (N, g_1)$ is a Riemannian submersion with fundamental tensor

$$(2.11) \quad T_U V = -\frac{1}{2f}g(U, V)\text{grad}f.$$

The above formula shows that the fibres become totally umbilical submanifolds of $M \times N$. The associated mean curvature vector field $H = -\frac{1}{2f}\text{grad}f$ shows further that the fibres cannot be minimal (equivalently, since umbilical, nor geodesic) unless

f is constant. If f is constant, one essentially gets again the product manifold of the above example, where both projections are Riemannian submersions with totally geodesic fibres.

Example 2.6. Take as π the natural projection of the tangent bundle of a Riemannian manifold (M, g) to it. Then TM can be endowed with the canonical metric (Sasakian metric) G by:

$$G(\tilde{X}, \tilde{Y}) = g(\pi_*\tilde{X}, \pi_*\tilde{Y}) \circ \pi + g(K\tilde{X}, K\tilde{Y}) \circ \pi$$

where K is the Dombrowski connection map associated to ∇^M . One immediately gets the Riemannian submersion definition properties for $\pi : (TM, G) \rightarrow (M, g)$. The fibres are totally geodesic. If TM is endowed with the Cheeger-Gromoll metric (see [2]) one gets a Riemannian submersion from (TM, \tilde{g}) to (M, g) , as well.

Example 2.7. Taking above the hypersurface consisting of unit vectors in TM with respect to the metric g , one gets that $T_1M = \{v \in TM \mid \|v\| = 1\}$ is the total space (of a spherical bundle over the base manifold M) with the almost contact metric structure naturally induced by the almost Hermitian structure on TM and the projection $\pi : T_1M \rightarrow M$ is a contact-complex Riemannian submersion [2].

Example 2.8. The Hopf fibration $\pi : S^{2n+1} \rightarrow CP^n$ gives a contact-complex Riemannian submersion considering the Sasakian structure on S^{2n+1} and a suitable multiple of the Fubini-Study metric on the complex manifold CP^n .

3. TENSOR B OF CONTACT-COMPLEX RIEMANN SUBMERSIONS

Define on the total space a $(1, 2)$ -tensor B in the contact-complex Riemannian submersions setting by

$$(3.1) \quad B(\tilde{X}, \tilde{Y}) = v\nabla_{h\tilde{X}}\varphi h\tilde{Y} - v\nabla_{\varphi h\tilde{X}}h\tilde{Y} + h\nabla_{h\tilde{X}}\varphi v\tilde{Y} - h\nabla_{\varphi h\tilde{X}}v\tilde{Y},$$

where ∇ is the Levi-Civita connection on (M, g) .

This tensor appears in the work of Watson and Vanhecke [35] on Riemannian submersions between almost Hermitian manifolds.

Proposition 3.1. *Let $\pi : M \rightarrow N$ be a contact-complex Riemannian submersion. Then*

- i. $B(V, \tilde{Y}) = 0$ for vertical V and arbitrary \tilde{Y} . In particular, B restricts to zero on the vertical distribution.
- ii. $B(\varphi X, \xi) = h\nabla_X \xi$ for any horizontal X . In particular, if M is Sasakian, then $B(\varphi X, \xi)$ is not zero. If M is almost cosymplectic, then $B(\varphi X, \xi) = 0$.
- iii. B restricts to a J -invariant tensor on the horizontal distribution \mathcal{H} .

Proof. i. The first assertion is trivial.

ii. From (3.1) one gets

$$B(\varphi X, \xi) = -h\nabla_{\varphi h\varphi X} \xi = -h\nabla_{\varphi^2 X} \xi = h\nabla_X \xi$$

since $\varphi^2 X = -X$, for any horizontal X . If M is Sasaki, then $\nabla_X \xi = -\varphi X$, so $B(\varphi X, \xi) \neq 0$ for any horizontal vector field X .

If M is cosymplectic we have $\nabla_X \xi = 0$ and the conclusion is obvious.

iii. Since \mathcal{H} is φ -invariant and $\varphi^2 X = -X$ for any horizontal vector field X , the result follows. \square

Proposition 3.2. *Let $\pi : M \longrightarrow N$ be a contact-complex Riemannian submersion. Then:*

- i. *For any vector field \tilde{Z} on M , the linear mapping $B(\tilde{Z}, \cdot)$ sends forward and backwards horizontal vectors to vertical vectors.*
- ii. *$B(X, Y) = B(Y, X) = A_X \varphi Y - A_{\varphi X} Y$ for horizontal vector fields X and Y .*
- iii. *$B(X, Y) = 0$ for horizontal X and Y if and only if $B(X, X) = 0$ for any horizontal X .*

Proof. i,ii. Trivial; iii. This statement follows immediately from the symmetry of B on the horizontal distribution \mathcal{H} . \square

Proposition 3.3. *Let $\pi : M \longrightarrow N$ be a contact-complex Riemannian submersion. Then for any basic X , Y and vertical V one has*

$$(3.2) \quad g(B(X, Y), V) = g(\nabla_V \varphi Y + \varphi \nabla_V Y, X) + 2Vg(\varphi X, Y)$$

Proof. Using the properties of A and φ one gets

$$\begin{aligned} g(A_X \varphi Y, V) &= g(\nabla_X \varphi Y, V) = -g(\varphi Y, \nabla_X V) = \\ &= -g(\varphi Y, [X, V] + \nabla_V X) = -g(\varphi Y, \nabla_V X) = \\ &= -Vg(X, \varphi Y) + g(X, \nabla_V \varphi Y) \end{aligned}$$

since the bracket $[X, V]$ is vertical. Analogously

$$g(\nabla_{\varphi X} Y, V) = -Vg(\varphi X, Y) - g(\varphi \nabla_V Y, X)$$

so that with property ii. in Proposition 3.2 the result follows. \square

Next, we recall the fundamental properties of Riemann contact-complex submersions due to Watson [34].

Proposition 3.4. [34] *Let $\pi : M \rightarrow N$ be a contact-complex Riemannian submersion. Then:*

- i. $\xi \in \Gamma(\mathcal{V}), \eta(X) = 0$, for any horizontal X ; the vertical and horizontal distributions \mathcal{V} and \mathcal{H} are φ -invariant.
- ii. $h(\nabla_X \varphi)Y$ is basic, π -related to $(\nabla'_{X'} J)Y'$, for basic X, Y on M , π -related to X', Y' on N .
- iii. $\pi^* \Omega = \Phi$.
- iv. $N_1(X, Y) = (\pi^* N_J)(X, Y)$, for basic X, Y , where N_1 denotes the normality tensor of M and N_J the Nijenhuis tensor of N .

Next, we investigate the contact-complex submersions satisfying Gray-type curvature conditions, denoted $K_{i\varphi}$, $i = 1, 2, 3$. We recall that the so-called Gray identities were introduced by A. Gray [14] for almost Hermitian manifolds (M, g, J) :

$$K_1 : R(X, Y, Z, W) = R(X, Y, JZ, JW),$$

$$K_2 : R(X, Y, Z, W) = R(JX, Y, Z, JW) + R(X, JY, Z, JW) + R(X, Y, JZ, JW),$$

$$K_3 : R(X, Y, Z, W) = R(JX, JY, JZ, JW).$$

For an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, the analogous identities have been considered by Bonome, Hervella and Rozas [5]:

$$K_{1\varphi} : R(X, Y, Z, W) = R(X, Y, \varphi Z, \varphi W),$$

$$K_{2\varphi} : R(X, Y, Z, W) = R(\varphi X, Y, Z, \varphi W) + R(X, \varphi Y, Z, \varphi W) + R(X, Y, \varphi Z, \varphi W),$$

$$K_{3\varphi} : R(X, Y, Z, W) = R(\varphi X, \varphi Y, \varphi Z, \varphi W).$$

We remark that other Gray-type curvature conditions on almost contact metric manifolds have been introduced in [21].

Proposition 3.5. *Let $\pi : M \rightarrow N$ a contact-complex submersion where the total space M satisfies $K_{1\varphi}$. Then:*

- i. *If O'Neill tensor T is φ -bilinear, then the fibres satisfy on the $K_{1\varphi}$.*
- ii. *If O'Neill tensor A vanishes, then the base space N satisfies on K_1 .*

Proof. i. The curvature relations in a Riemannian submersion setting give:

$$R(U, V, F, W) = \hat{R}(U, V, F, W) + g(T_U W, T_V F) - g(T_V W, T_U F),$$

$$R(U, V, \varphi F, \varphi W) = \hat{R}(U, V, \varphi F, \varphi W) + g(T_U \varphi W, T_V \varphi F) - g(T_V \varphi W, T_U \varphi F),$$

where R and \hat{R} are the curvature tensors on M and one of the fibres, respectively, U, V, F, W being vertical vector fields.

Then the assertion follows using the φ -invariance of the metric g .

ii. From

$$R(X, Y, Z, H) = R^*(X, Y, Z, H) - 2g(A_X Y, A_Z H) + g(A_Y Z, A_X H) - g(A_X Z, A_Y H),$$

where $R^*(X, Y, Z, H) = R'(X', Y', Z', H') \circ \pi$ and X, Y, Z, H are basic vector fields π -related to X', Y', Z', H' respectively, it follows that

$$\begin{aligned} R(X, Y, \varphi Z, \varphi H) &= R^*(X, Y, \varphi Z, \varphi H) - 2g(A_X Y, A_{\varphi Z} \varphi H) + \\ &+ g(A_Y \varphi Z, A_X \varphi H) - g(A_X \varphi Z, A_Y \varphi H). \end{aligned}$$

If $A \equiv 0$, then

$$R'(X', Y', Z', H') = R'(X', Y', JZ, JH)$$

since $\varphi Z, \varphi H$ are related to JZ' and JH' . \square

Proposition 3.6. *Let $\pi : M \rightarrow N$ be a contact-complex Riemannian submersion. Suppose that both M and N satisfy $K_{1\varphi}$ and K_1 respectively. If $[X, \varphi X] \in \Gamma(V)$ for any basic vector field X , then \mathcal{H} is integrable.*

Proof. The curvature equations give now, setting $Z = X$ and $H = Y$:

$$\begin{aligned} &-2g(A_X Y, A_X Y) + g(A_Y X, A_X Y) + g(A_X X, A_Y Y) + \\ &+ 2g(A_X Y, A_X Y) - g(A_Y \varphi X, A_X \varphi Y) - g(A_{\varphi X} X, A_Y \varphi Y) = 0. \end{aligned}$$

Because $[X, \varphi X] \in \Gamma(V)$ says that $B(X, X) = 0$ for an horizontal vector field X , it is now known that $B(X, Y) = 0$ for horizontal X, Y .

It easy to see that $A_{\varphi X} \varphi Y = -A_X Y$ and this gives now

$$5g(A_X Y, A_X Y) + g(A_X \varphi Y, A_X \varphi Y) = 0.$$

Both summands are non-negative, so that vanish; in particular \mathcal{H} is involutive. \square

Proposition 3.7. *Let $\pi : M \rightarrow N$ be a contact-complex Riemannian submersion. Suppose that M satisfies on the $K_{2\varphi}$. If A is φ -bilinear then the base space N also satisfies on K_2 .*

Proof. The fundamental equations of a submersion give, taking into account the hypothesis, that

$$\begin{aligned} &R'(X', Y', Z', T') - R'(JX', JY', Z', T') - R'(JX', Y', JZ', T') - R'(JX', Y', Z', JT') = \\ &= 2g(A_X Y, A_Z T) - g(A_Y Z, A_X T) - g(A_Z X, A_Y T) \\ &+ g(A_{\varphi Y} Z, A_{\varphi X} T) - 2g(A_{\varphi X} \varphi Y, A_Z T) + g(A_Z \varphi X, A_{\varphi Y} T) \end{aligned}$$

$$\begin{aligned}
& +g(A_Y\varphi Z, A_{\varphi X}T) - 2g(A_{\varphi X}Y, A_{\varphi Z}T) + g(A_{\varphi Z}\varphi X, A_YT) \\
& +g(A_YZ, A_{\varphi X}\varphi T) - 2g(A_{\varphi X}Y, A_Z\varphi T) + g(A_Z\varphi X, A_Y\varphi T).
\end{aligned}$$

The right-hand side vanishes as follows. We have

$$g(A_{\varphi Y}Z, A_{\varphi X}T) = g(\varphi A_YZ, \varphi A_XT) = g(A_YZ, A_XT) + \eta(A_YZ)\eta(A_XT),$$

since A is φ -bilinear. On the other side,

$$\eta(A_YZ) = g(A_YZ, \xi) = g(A_Y\varphi P, \xi) = g(\varphi A_YP, \xi) = 0$$

taking into account that \mathcal{H} is φ -invariant, as well, so that it exists $P \in \Gamma(\mathcal{H})$ such that $Z = \varphi P$. Then $g(A_{\varphi Y}Z, A_{\varphi X}T) = g(A_YZ, A_XT)$.

Now consider $-2g(A_{\varphi X}\varphi Y, A_ZT)$ which can be expanded

$$\begin{aligned}
g(A_{\varphi X}\varphi Y, A_ZT) &= g(\varphi^2 A_XY, A_ZT) = \\
&= -g(A_XY, A_ZT) + \eta(A_XY)g(\xi, A_ZT) = -g(A_XY, A_ZT).
\end{aligned}$$

In an analogous way, all terms containing φ will be replaced by quantities not involving φ so that

$$R'(X', Y', Z', T') = R'(JX', JY', Z', T') + R'(JX', Y', JZ', T') + R'(JX', Y', Z', JT')$$

for any vector fields X', Y', Z', T' on the base space N . \square

One gets in an analogous way the following result.

Proposition 3.8. *Let $\pi : M \rightarrow N$ be a contact-complex Riemannian submersion. Suppose that the O'Neill tensor A satisfies on $A_X\varphi Y = A_{\varphi X}Y$, for any horizontal vector fields X, Y . If $K_{3\varphi}$ holds on M , then K_3 holds on N .*

4. THE STRUCTURE EQUATION OF A CONTACT-COMPLEX RIEMANNIAN SUBMERSION

The relationship between the codifferentials of the basic 2-forms Φ , $\hat{\Phi}$ and Ω of the total space, a fibre and the base space of a contact-complex Riemannian submersion $\pi : M \rightarrow N$ will be established in what follows.

The codifferential of Φ on M is

$$\begin{aligned}
\delta\Phi(\tilde{X}) &= - \sum_{i=1}^n \{(\nabla_{X_i}\Phi)(X_i, \tilde{X}) + (\nabla_{\varphi X_i}\Phi)(\varphi X_i, \tilde{X})\} \\
(4.1) \quad &- \sum_{j=1}^{m-n} \{(\nabla_{V_j}\Phi)(V_j, \tilde{X}) + (\nabla_{\varphi V_j}\Phi)(\varphi V_j, \tilde{X})\} - (\nabla_{\xi}\Phi)(\xi, \tilde{X})
\end{aligned}$$

with

$$\{(X_i, \varphi X_i); (V_i, \varphi V_i, \xi)\}, 1 \leq i \leq n, 1 \leq j \leq m-n,$$

a φ -orthonormal local basis in the tangent space, such that $(X_i, \varphi X_i)$ are basic and $(V_i, \varphi V_i)$ are vertical. Recall that $\varphi X_i, \varphi V_i$ are horizontal and vertical respectively, since both distributions \mathcal{H} , \mathcal{V} are φ -invariant, and the Reeb vector field ξ is vertical.

Theorem 4.1. (The structure equation) *Let $\pi : M \rightarrow N$ be a contact-complex Riemannian submersion. The following formula holds good:*

$$(4.2) \quad \delta\Phi(\tilde{X}) = \delta'\Omega(X') + \hat{\delta}\hat{\Phi}(V) + g(H, \varphi X) + \frac{1}{2}g(\text{Tr}B^h, V)$$

where $\delta, \hat{\delta}, \delta'$ are the codifferential on M , N and a fibre (by respect to the induced metric). Here H is the mean curvature vector field of a fibre and $Tr B^h$ stands for the trace of the restriction of the tensor B to the distribution \mathcal{H} .

Proof. If \tilde{X} is an arbitrary vector field on M , then there is an unique decomposition $\tilde{X} = X + V$ where X is horizontal and V is vertical, so that $X = h\tilde{X}$ and $V = v\tilde{X}$. One calculates next $\delta\Phi(X)$ and $\delta\Phi(V)$. Using the formula

$$(\nabla_{\tilde{X}}\Phi)(\tilde{Y}, \tilde{Z}) = -g((\nabla_{\tilde{X}}\varphi)\tilde{Y}, \tilde{Z})$$

it follows from (4.1) that

$$\begin{aligned} \delta\Phi(X) &= \sum_{i=1}^n \{g((\nabla_{X_i}\varphi)X_i, X) + g((\nabla_{\varphi X_i}\varphi)X_i, X)\} \\ &\quad + \sum_{j=1}^{m-n} \{g((\nabla_{V_j}\varphi)V_j, X) + g((\nabla_{\varphi V_j}\varphi)V_j, X)\} + g((\nabla_{\xi}\varphi)\xi, X). \end{aligned}$$

The generic term in the first summation can be written as

$$g(\nabla_{X_i}\varphi X_i - \varphi\nabla_{X_i}X_i, X) - g(\nabla_{\varphi X_i}X_i, X) - g(\varphi\nabla_{\varphi X_i}\varphi X_i, X)$$

or

$$g((h\nabla_{X_i}\varphi X_i, X) - g(h\varphi\nabla_{X_i}X_i, X) - g(h\nabla_{\varphi X_i}X_i, X) - g(h\varphi\nabla_{\varphi X_i}\varphi X_i, X).$$

Considering now basic, local vector fields $X_i, \varphi X_i, X$ on M , π -related to $X'_i, \varphi X'_i, X'$ on N , we obtain

$$g'((\nabla'_{X'_i}JX'_i, X') - g'(J\nabla'_{X'_i}X'_i, X') - g'(\nabla'_{JX'_i}X'_i, X') - g'(J\nabla'_{JX'_i}JX'_i, X')).$$

Therefore the first sum in the expression of $\delta\Phi(X)$ can be written as

$$\sum_{i=1}^n \{g'((\nabla'_{X'_i}J)X'_i, X')o\pi + g'((\nabla'_{JX'_i}J)JX'_i, X')o\pi\}$$

or, equivalently, $\delta'\Omega(X')o\pi$, where Ω is the fundamental form on the base space N .

Next, the generic summand in the expression of $\delta\Phi(X)$ (the second summation) is

$$g(\nabla_{V_j}\varphi V_j - \varphi\nabla_{V_j}V_j, X) - g(\nabla_{\varphi V_j}V_j + \varphi\nabla_{\varphi V_j}\varphi V_j, X)$$

or

$$g(\nabla_{V_j}\varphi V_j - \nabla_{\varphi V_j}V_j, X) - g(\varphi(\nabla_{V_j}V_j + \nabla_{\varphi V_j}\varphi V_j), X).$$

The first term above vanishes as $g([V_j, \varphi V_j], X) = 0$, since $[V_j, \varphi V_j]$ is tangent to the fibres. Hence the second sum in the expression of $\delta\Phi(X)$ can be written as

$$\sum_{i=1}^n g(\nabla_{V_j}V_j + \nabla_{\varphi V_j}\varphi V_j, \varphi X),$$

or, equivalently, $g(H, \varphi X) - g(\nabla_{\xi}\xi, \varphi X)$, where H is the mean curvature vector field of the fibres.

One finally gets therefore

$$(4.3) \quad \delta\Phi(X) = g(H, \varphi X) + \delta'\Omega(X')o\pi.$$

On another hand, we have:

$$\begin{aligned}\delta\Phi(V) &= \sum_{i=1}^n \{g((\nabla_{X_i}\varphi)X_i, V) + g((\nabla_{\varphi X_i}\varphi)X_i, V)\} \\ &+ \sum_{j=1}^{m-n} \{g((\nabla_{V_j}\varphi)V_j, V) + g((\nabla_{\varphi V_j}\varphi)V_j, X)\} + g((\nabla_{\xi}\varphi)\xi, V).\end{aligned}$$

For the generic summand in the first summation we obtain

$$\begin{aligned}g((v\nabla_{X_i}\varphi X_i - v\varphi\nabla_{X_i}X_i, V) - g(v\nabla_{\varphi X_i}X_i + v\varphi\nabla_{\varphi X_i}\varphi X_i, V) \\ = g(A_{X_i}\varphi X_i - A_{\varphi X_i}X_i, V) - g(\varphi A_{X_i}X_i + \varphi A_{\varphi X_i}\varphi X_i, V)\end{aligned}$$

by the definition of A and the fact that φ and the projection v commute. The skew-symmetry of A gives the vanishing of the last parenthesis.

Denoting by B^h the restriction of B to the horizontal distribution, it follows that

$$g(A_{X_i}\varphi X_i - A_{\varphi X_i}X_i, V)$$

is the generic summand in $\frac{1}{2}g(TrB^h, V)$, where TrB^h is the trace of B^h . Compute now the second sum in $\delta\Phi(V)$ as the restriction of $\delta\Phi(V)$ to the vertical distribution \mathcal{V} . Distinguish by a hat the induced objects on fibres (which are φ -invariant submanifolds). It follows that

$$\hat{\delta}\hat{\Phi}(V) = - \sum_{j=1}^{m-n} \{(\hat{\nabla}_{V_j}\hat{\Phi})(V_j, V) + (\hat{\nabla}_{\varphi V_j}\hat{\Phi})(\varphi V_j, V)\} - (\hat{\nabla}_{\xi}\hat{\Phi})(\xi, V).$$

The first summand expands

$$\begin{aligned}(\hat{\nabla}_{V_j}\hat{\Phi})(V_j, V) &= -g((\hat{\nabla}_{V_j}\varphi)V_j, V) = -g(\hat{\nabla}_{V_j}\varphi V_j, V) + g(\varphi\hat{\nabla}_{V_j}V_j, V) \\ &= -g(\nabla_{V_j}\varphi V_j - \varphi\nabla_{V_j}V_j, V) = -g((\nabla_{V_j}\varphi)V_j, V) \\ &= (\nabla_{V_j}\Phi)(V_j, V).\end{aligned}$$

In an analogous way, it follows that $\delta\Phi(V)$ restricts to the fibres to $\hat{\delta}\hat{\Phi}(V)$. Therefore

$$(4.4) \quad \delta\Phi(V) = \hat{\delta}\hat{\Phi}(V) + \frac{1}{2}g(TrB^h, V)$$

From (4.3) and (4.4) one gets (4.2), where $\delta\Phi(\tilde{X}) = \delta\Phi(X) + \delta\Phi(V)$. \square

In what follows this formula will be used in the study of almost cosymplectic manifolds with Kähler fibres introduced and studied by Z. Olszak [25].

5. CONTACT-COMPLEX RIEMANNIAN SUBMERSIONS FROM ALMOST COSYMPLECTIC MANIFOLDS WITH KÄHLER LEAVES

Recall that an almost contact metric manifold M is almost cosymplectic if η and the fundamental form Φ are closed, i.e.

$$(5.1) \quad d\Phi = 0, d\eta = 0.$$

The identity $d\eta = 0$ shows that the distribution $D = \{X \in TM \mid \eta(X) = 0\}$ is integrable and its (maximal) integral manifolds are hypersurfaces in M . The restrictions of Φ and η to the associated foliation are closed forms, so that any leave is an almost Kähler submanifold.

Denote by A^* the endomorphism of the tangent space given by

$$(5.2) \quad A^*X = -\nabla_X \xi, \quad \forall X \in \Gamma(TM).$$

Proposition 5.1. [25] *The tensor A^* satisfies on the following formulae*

$$(5.3) \quad g(A^*X, Y) = g(X, A^*Y),$$

$$(5.4) \quad A^*\varphi + \varphi A^* = 0, \quad A^*\xi = 0, \quad \eta \circ A^* = 0,$$

$$(5.5) \quad (\nabla_X \varphi)Y = -g(\varphi A^*X, Y)\xi + \eta(Y)\varphi A^*X.$$

It is trivial that a cosymplectic manifold is almost cosymplectic with Kähler leaves. Also, the tensor field A^* vanishes in this case. Olszak gave examples of almost cosymplectic manifolds with Kähler leaves that are not cosymplectic (see [25]). He proved as well that if an almost cosymplectic manifold with Kähler leaves has vanishing tensor A^* , then it is actually cosymplectic.

We recall that an almost contact structure (ϕ, ξ, η) is normal if and only if the next four tensors N_1, N_2, N_3 and N_4 vanish (see [2]):

$$N_1(X, Y) = [\phi, \phi](X, Y) + 2d\eta(X, Y)\xi,$$

$$N_2(X, Y) = (L_{\varphi X}\eta)Y - (L_{\varphi Y}\eta)X,$$

$$N_3(X) = (L_\xi \varphi)X,$$

$$N_4(X) = (L_\xi \eta)X,$$

where L_Z denotes the Lie derivative with respect to Z . It is known that the vanishing of N_1 on contact metric manifolds gives the vanishing of N_2, N_3 and N_4 , too. The converse is not true, in general (see [2] for details). The almost cosymplectic manifold with Kähler leaves have, generally, nonvanishing tensors N_1 and N_3 , but N_2 and N_4 are all zero, as we can see in what follows.

Proposition 5.2. *Let M be an almost cosymplectic manifold with Kähler leaves. Then N_2 and N_4 vanish. Moreover N_3 vanishes if and only if M is a cosymplectic manifold.*

Proof. The vanishing of N_2 and N_4 follows easily using (5.3), (5.4) and (5.5).

On the other hand, by direct computation we derive

$$\begin{aligned} N_3(X) &= (L_\xi \varphi)X = [\xi, \varphi X] - \varphi[\xi, X] \\ &= \nabla_\xi \varphi X - \nabla_{\varphi X} \xi - \varphi \nabla_\xi X + \varphi \nabla_X \xi \\ &= A^* \varphi X - \varphi A^* X \\ &= 2A^* \varphi X \end{aligned}$$

and the conclusion follows. \square

Proposition 5.3. *Let $M(\varphi, \xi, \eta, g)$ be an almost cosymplectic manifold with Kähler leaves. Then its fundamental form is harmonic.*

Proof. Since Φ is closed, it remains to prove that it is co-closed, as well.

Let $\{e_i, \varphi e_i, \xi\}$ be a local φ -Hermitian frame on M , ($1 \leq i \leq m$). It follows that

$$\delta \Phi(\tilde{X}) = -\sum_{i=1}^m (\nabla_{e_i} \Phi)(e_i, \tilde{X}) - \sum_{i=1}^m (\nabla_{\varphi e_i} \Phi)(\varphi e_i, \tilde{X}) - (\nabla_\xi \Phi)(\xi, \tilde{X})$$

which can be written

$$\delta\Phi(\tilde{X}) = \sum_{i=1}^m g((\nabla_{e_i}\varphi)e_i, \tilde{X}) + \sum_{i=1}^m g((\nabla_{\varphi e_i}\varphi)\varphi e_i, \tilde{X}) + g((\nabla_{\xi}\varphi)\xi, \tilde{X}).$$

The last summand is obviously zero. Taking into account (5.5), the generic summand in the first summation takes the form

$$g((\nabla_{e_i}\varphi)e_i, \tilde{X}) = -g(\varphi A^*(e_i), e_i)g(\xi, \tilde{X}).$$

The second term is

$$g((\nabla_{\varphi e_i}\varphi)\varphi e_i, \tilde{X}) = -g(\varphi A^*(\varphi e_i), \varphi e_i)g(\xi, \tilde{X}).$$

Using the properties of A^* , it follows that $\delta\Phi(\tilde{X}) = 0$, $\forall \tilde{X} \in \Gamma(TM)$. Finally

$$\Delta\Phi = (d\delta + \delta d)\Phi = 0$$

and the result follows. \square

Theorem 5.4. *Let $\pi : M \rightarrow N$ be a contact-complex Riemannian submersion. Suppose that M is almost cosymplectic with Kähler leaves. Then*

- i. The base space N is Kähler if and only if the O'Neill tensor A satisfies $A_X\xi = 0$ for any $X \in \mathcal{H}$.*
- ii. The fundamental form Ω of the base is harmonic if and only if $\text{Tr}B^h = 0$.*

Proof. i. From Proposition 3.4 one has

$$(\pi^*N_j)(X, Y) = N_1(X, Y)$$

for any basic vector fields X, Y on M .

One shows next that $N_1(X, Y) = 0$ for any basic vector fields X, Y . It is known that the vanishing of N_1 is equivalent to the identity

$$(\nabla_X\varphi)Y - (\nabla_{\varphi X}\varphi)\varphi Y + \eta(Y)\nabla_{\varphi X}\xi = 0.$$

But the last term vanishes as Y is basic. Using (5.5), the last equation becomes

$$g(\varphi\nabla_X\xi, Y) - g(\varphi\nabla_{\varphi X}\xi, \varphi Y) = 0$$

or, equivalently, $(\nabla_X\xi, \varphi Y) = 0$. Taking into account that $A_X\xi = h\nabla_X\xi$ and the φ -invariance of \mathcal{H} , the assertion follows as

$$N_1(X, Y) = N_J(X', Y'),$$

with X, Y basic vector fields on M , π -related to X', Y' on N .

- ii. It is known that a φ -invariant submanifold of an almost cosymplectic manifold tangent to the characteristic vector field ξ is minimal, so that the fibres are minimal: $H = 0$. The restriction to the fibres of δ is $\hat{\delta}$, so that $\hat{\delta}\hat{\Phi} = 0$.

Since $\delta\Phi = 0$, the assertion follows from the structure equation (4.2). \square

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